Algebraic structures for 1D quasiperiodic systems

1993 J. Phys. A: Math. Gen. 26 213

(http://iopscience.iop.org/0305-4470/26/2/010)

J. Phys. A: Math. Gen. 26 (1993) 213–228. Printed in the UK

Algebraic structures for one-dimensional quasiperiodic systems

Peter Kramer
Institut für Theoretische Physik, Auf der Universität Tübingen, Morgenstelle 14, D-7400 Tübingen, Federal Republic of Germany

Received 26 May 1992, in final form 2 September 1992

Abstract. The structure of certain quasiperiodic systems is analysed and interpreted by algebraic means. All systems considered have a free monoid and semigroup structure with n generators, as treated in the field of combinatorics on words. An extension from free monoids to free groups and their automorphisms, as treated by combinatorial group theory, applies to a class of iterative systems. The group structures yield algebraic relations and conserved quantities. Examples encompass the Fibonacci system and its generalizations. Motivated by the applications to the physics of these systems, homomorphisms from the free groups into linear operators of SU(2) and SU(1, 1) are studied. The systems are also viewed in a geometry due to Fricke and Klein. The homomorphism to the group SU(1, 1) is the basis for the determination of 1D electronic spectra. The class types, their multiplication, and the commutator properties are given. The physics of these systems corresponds to the continuous Schrödinger equation. The quasiperiodic structure gives rise to exact relations between continuous and discrete systems.

1. Free monoids, free groups and automorphisms

Sequences like the Fibonacci or Thue–Morse system may be viewed in the field of combinatorics on words. We refer to Lothaire [11]: Given a (finite) set called an alphabet $A = (x_1, \ldots, x_n)$, any finite sequence formed from elements of A is called a word w. The set of all words is denoted by A^*. The empty word e is defined by $ew = we = w$. If the multiplication of words is defined by concatenation, the set A^* including the empty word becomes a free monoid and the set $A^+ = A^* - e$ a free semigroup. A morphism ϕ from a monoid A^* to another monoid A'^* is defined by the properties $\phi(w_1 w_2) = \phi(w_1)\phi(w_2)$ for any two words $w_1, w_2 \in A^*$. To specify a morphism it clearly suffices to give the images of the alphabet A.

Given the monoid with alphabet A, one can formally introduce inverses with the properties $x_i^{-1} : x_i^{-1}x_i = x_i x_i^{-1} = e, 1 \leqslant i \leqslant n$, compare Lothaire [11]. Now the alphabet is called the generating set. The free group generated from it is denoted by F_n. We continue to call the elements of F_n words. An automorphism ρ is an invertible morphism $\rho : F_n \to F_n$. The set of all automorphisms of the free group F_n is a group denoted by Φ_n. This group was characterized in general by Nielsen, see Magnus [12].

The Fibonacci strings may be interpreted as a particular set of words from A^* based on an alphabet $A : (x_1, x_2)$. Consider the standard local inflation rule, written in a two-line notation as

$$\begin{array}{rll} & x_1 & x_2 \\ \phi : & x_2 & x_1 x_2 \end{array} \tag{1}$$

which by iteration generates from suitable initial words the Fibonacci string. Extend the monoid A^* to the free group F_2. Observe that the map ϕ has an inverse

$$\begin{array}{rll} & x_1 & x_2 \\ \phi^{-1} : & x_2 x_1^{-1} & x_1 \end{array} \tag{2}$$

and, by extension to all words, becomes an automorphism of F_2. This allows us to give a new interpretation of the Fibonacci chain in terms of F_2: the (infinite) Fibonacci chain is *the orbit starting at x_1 under the positive powers of the automorphism ϕ*. The substitution matrices for all automorphisms define a homomorphism $\Phi_2 \to \mathrm{Gl}(2, Z)$.

Note that substitution rules like the one for the Thue–Morse system (Lothaire [11], section 2.2) are not invertible. This suggests a separate study of the class of iterative systems which form orbits under a subgroup of automorphisms of the free group. For all (invertible) automorphisms of F_2 we have the following theorem.

Theorem (Nielsen 1918). Let F_2 be the free group with two generators x_1, y_2. Let $\rho(x_1), \rho(x_2)$ be the maps of x_1, x_2 under any automorphism $\rho \in F_2$. Then the commutator under ρ transforms as

$$\rho(x_1 x_2 x_1^{-1} x_2^{-1}) = w(x_1 x_2 x_1^{-1} x_2^{-1})^{\pm 1} w^{-1} \tag{3}$$

where w is a word from F_2.

For a proof, it suffices to verify the theorem for the generators of Φ_2 given by Nielsen, see Magnus [12]. In the following equation (4) we give Nielsen's generators P, σ, U of Φ_2 along with the image of $(x_1 x_2)^{-1}$, in the notation of Magnus [12] and the corresponding element of $\mathrm{Gl}(2, Z)$:

$$\begin{array}{rllll} & x_1 & x_2 & x_2^{-1} x_1^{-1} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ P : & x_2 & x_1 & x_1^{-1} x_2^{-1} & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma : & x_1^{-1} & x_2 & x_2^{-1} x_1 & \\ U : & x_1 x_2 & x_2 & x_2^{-2} x_1^{-1} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \end{array} \tag{4}$$

Nielsen and Magnus give the relations fulfilled by the generators of Φ_2. The Fibonacci case equation (1) is generated as $\phi = P \circ U$. Next consider a particular set of systems based on F_2: Let ρ be an endomorphism

$$\begin{array}{rll} & x_1 & x_2 \\ \rho : & x_2 & f(x_1, x_2). \end{array} \tag{5}$$

Then ρ is an automorphism iff there is a function h,

$$x_3 := f(x_1, x_2) \leftrightarrow x_1 = h(x_2, x_3). \tag{6}$$

For the proof we note that the endomorphism

$$\begin{array}{lll} & x_1 & x_2 \\ \rho^* : & h(x_1, x_2) & x_1 \end{array} \tag{7}$$

is inverse to ρ.

For ρ with this property, the automorphism ρ^m has the two-step recursion relation

$$\rho^{n+1}(x_2) = f(\rho^{n-1}(x_2), \rho^n(x_2)). \tag{8}$$

These relations follow from

$$\begin{aligned} \rho^n(x_1) &= \rho^{n-1}(x_2) \\ \rho^{n+1}(x_2) &= f(\rho^n(x_1), \rho^n(x_2)) \\ &= f(\rho^{n-1}(x_2), \rho^n(x_2)). \end{aligned} \tag{9}$$

In fact it is easy to construct explicitly all the automorphisms ρ. They are of the form

$$\begin{array}{lll} & x_1 & x_2 \\ \rho : & x_2 & x_2^p x_1 x_2^{m-p} \end{array} \qquad m, p \in Z \tag{10}$$

and the map to $\mathrm{Gl}(2, Z)$ yields

$$\rho \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}. \tag{11}$$

For applications to quasicrystals we require, in addition, that all powers be non-negative, hence $p \geqslant 0, m - p \geqslant 0$ or $m > 0, p = 0, \ldots, m$. This class of systems will be called generalized Fibonacci systems. The theorem by Nielsen applies to all these systems, and one easily obtains for the commutator $K = x_1 x_2 x_1^{-1} x_2^{-1}$

$$\rho(K) = w K^{-1} w^{-1} \qquad w = x_2^p. \tag{12}$$

Note that for $p = 0$ the commutator is an improper invariant, it becomes an invariant for ρ^2. Clearly there is a wider class of systems, generated by powers of a fixed automorphism, to which the Nielsen theorem applies. In sections 2–4 we study homomorphisms $F_2 \rightarrow \mathrm{SU}(2), F_2 \rightarrow \mathrm{SU}(1,1)$, develop the appropriate algebra and geometry of these groups, and give the transformations of group elements generated by automorphisms from Φ_2. In sections 5–7 we relate these structures to the physics of the Fibonacci and related systems. Applications in physics with this group structure were given in [14], [10] and [7], a detailed list of references is given in [1].

2. Classes and commutators of the group SU(1,1)

The Lie group $\mathrm{SU}(1,1)$ (compare Bargmann [2]) has as its elements all complex 2×2 matrices

$$T : TMT^{+} = M \qquad M = M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{13}$$

$$\det(T) = 1. \tag{14}$$

With the definition of the $\mathrm{U}(1,1)$-adjoint $Q \to Q^{*} := MQ^{+}M$, the first set of conditions becomes $TT^{*} = e$. Here $e = e_{\mathrm{SU}(1,1)}$ is the unit element of $\mathrm{SU}(1,1)$, but we shall suppress the subscript if no confusion can arise. With this adjoint, we shall introduce Hermitian and anti-Hermitian complex 2×2 matrices. The representation of the Lie algebra of $\mathrm{SU}(1,1)$ is given by all anti-Hermitian traceless matrices $A : A = -A^{*}, \mathrm{tr}(A) = 0$. The elements of $\mathrm{SU}(1,1)$ may then be written in terms of two complex numbers λ, μ, Bargmann [2], as

$$T : T = \begin{bmatrix} \lambda & \mu \\ \overline{\mu} & \overline{\lambda} \end{bmatrix} \qquad \lambda\overline{\lambda} - \mu\overline{\mu} = 1. \tag{15}$$

The two complex numbers can be expressed in terms of four real numbers according to $\lambda = \xi_0 + \mathrm{i}\xi_3, \mu = \xi_1 + \mathrm{i}\xi_2$. For the elements T of the group $\mathrm{SU}(1,1)$ we get in this notation

$$\begin{aligned} T &= \begin{bmatrix} \xi_0 + \mathrm{i}\xi_3 & \xi_1 + \mathrm{i}\xi_2 \\ \xi_1 - \mathrm{i}\xi_2 & \xi_0 - \mathrm{i}\xi_3 \end{bmatrix} \\ &= \sum_{\mu} \xi_\mu \sigma'_\mu \\ \det(T) &= \xi_0^2 + \xi_3^2 - \xi_1^2 - \xi_2^2 = 1 \\ 1 - \xi_0^2 &= \xi_3^2 - \xi_1^2 - \xi_2^2 \\ \tfrac{1}{2}\,\mathrm{tr}(T) &= \xi_0. \end{aligned} \tag{16}$$

Here the matrices, defined by $\sigma'_1 = \sigma_1, \sigma'_2 = -\sigma_2, \sigma'_3 = i\sigma_3$ in terms of the Pauli matrices, form a basis of the traceless anti-Hermitian matrices, and $\sigma'_0 = e$. The decomposition of T into a (Hermitian) multiple of the unit matrix and a (anti-Hermitian) traceless part is

$$\begin{aligned} T^{\pm 1} &= S \pm A \\ S &= S^{*} = \xi_0 \sigma'_0 \\ A &= -A^{*} = \sum_{j=1}^{3} \xi_j \sigma'_j. \end{aligned} \tag{17}$$

The trace of T is a class function and may be used to characterize the classes. In the coordinates ξ_μ, any matrix T determines a point on a fixed three-dimensional quadratic surface of R^4. It can be seen from equation (16) that, for fixed class and hence value ξ_0, the elements T correspond to the points on a (in general) two-dimensional quadratic surface of R^3 with the metric of SO$(2,1,R)$. In the following table we characterize types of classes C by a short symbol, indexed with signs that characterize ranges of significant quadratic expressions in the numbers ξ_μ, and with the sign of the trace ξ_0 in front. Intervals given with round brackets exclude the points at the boundary.

Symbol	$-1+\xi_0^2$	ξ_3^2	Sign(ξ_0)
$\pm C_+^-$	$(-1,0)$	$(0,\infty)$	± 1
$\pm C_0^0$	0	0	± 1
$\pm C_+^0$	0	$(0,\infty)$	± 1
$\pm C_+^+$	$(0,\infty)$	$(0,\infty)$	$\pm 1.$

(18)

The eigenvalues and standard forms for the various class types are as follows. Elements from $\pm C_+^-$ have two complex conjugate eigenvalues with absolute value smaller than 1. The single elements of $\pm C_0^0$ are $(\pm e)$ respectively, the elements from $\pm C_+^0$ have a standard real triangular Jordan form. The last two types cannot be distinguished by their traces alone. The elements from $\pm C_+^+$ have two real eigenvalues, one with absolute value larger than 1.

For later applications we wish to characterize the commutator

$$K(T_1,T_2) = T_1 T_2 T_1^{-1} T_2^{-1}$$

of two group elements in terms of the class type. The commutator obeys $K(T_2,T_1) = K^{-1}(T_1,T_2)$. To determine the class type of the commutator, it suffices to choose one of the group elements in a standard form. Under the replacement $T \to -T$, which is well defined for any matrix group, the commutator obeys the relations $K(-T_1,T_2) = K(T_1,-T_2) = K(T_1,T_2)$. In view of these relations, we list in the following table the computed class types which can arise from the commutator of pairs of elements chosen from the (positive) class types defined earlier:

$K(T_1,T_2), T_2 \to$ $T_1 \downarrow$	C_+^-	C_0^0	C_+^0	C_+^+
C_+^-	C_0^0, C_+^+	C_0^0	C_+^+	C_+^+
C_0^0	C_0^0	C_0^0	C_0^0	C_0^0
C_+^0	C_+^+	C_0^0	C_0^0, C_+^+	C_+^0, C_+^+
C_+^+	C_+^+	C_0^0	$C_+^0, C_+^+,$	$\pm C_+^-, +C_0^0, \pm C_+^0, \pm C_+^+.$

(19)

Since the interchange of the elements yields the inverse commutator, and since the inverse element belongs to the same class type, this table is symmetric with respect to its diagonal. Two elements can commute if one of them belongs to $\pm C_0^0$ or if both belong to the same class type. Note the selective properties given in the table: many types cannot occur from commuting fixed types, class types with negative trace appear only in the last entry, and the class $-C_0^0 = (-e)$ never occurs as a commutator. The group generated by all the commutators is an invariant subgroup, but must coincide with SU(1,1) since this group has no proper invariant subgroup except $(e, -e)$.

3. Fricke–Klein geometry of SU(2)

For the group SU(2) we shall introduce a particular setting which borrows ideas from Fricke and Klein [6]. We use an exponential parametrization and write for any element

$$
\begin{aligned}
g &= \exp(-\mathrm{i}\alpha \sum_{j=1}^{3} \eta_i \sigma_i) \\
&= (\cos\alpha)\sigma_0 - \mathrm{i}(\sin\alpha) \sum_{j=1}^{3} \eta_j \sigma_j .
\end{aligned}
\tag{20}
$$

where $\sigma_0 = e$, the vector η is a unit vector, and where the Pauli matrices obey

$$
\begin{aligned}
\sigma_l\sigma_j + \sigma_j\sigma_l &= 2\delta_{lj}\sigma_0 \\
\sigma_l\sigma_j - \sigma_j\sigma_l &= 2i\epsilon_{ljk}\sigma_k .
\end{aligned}
\tag{21}
$$

We shall use the scalar product corresponding to SO(3, R) in all vector computations. The group multiplication with these relations becomes

$$
\begin{aligned}
&g_1 g_2 = (\cos\alpha_{12})\sigma_0 - \mathrm{i}(\sin\alpha_{12}) \sum_{j=1}^{3} \eta_j^{12} \sigma_j \\
&\cos\alpha_{12} = \cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 (\eta^1 \cdot \eta^2) \\
&(\sin\alpha_{12})\eta^{12} = (\sin\alpha_1 \cos\alpha_2)\eta^1 + (\cos\alpha_1 \sin\alpha_2)\eta^2 + (\sin\alpha_1 \sin\alpha_2)(\eta^1 \times \eta^2).
\end{aligned}
\tag{22}
$$

This multiplication has the following interpretation in R^3. The vectors η^1, η^2 each determine a great circle on the unit sphere with an oriented, right-handed arc of length α_1, α_2. These two oriented arcs may be concatenated in the order first 2, then 1 in one of the two intersection points of the two great circles. Their join determines a new great circle with an oriented arc, which yields the product element $g_1 g_2$. We stress that this construction in R^3 describes the multiplication in SU(2), not the adjoint representation discussed later. We define g_3 by

$$
g_1 g_2 g_3 = e \tag{23}
$$

and consider the spherical triangle formed by the three vectors $\eta^i, i = 1,2,3$. Define the dual unit vectors $\xi^i, i = 1,2,3$ by

$$
\xi^i = \epsilon_{ikl}(\eta^k \times \eta^l)|\eta^k \times \eta^l|^{-1}. \tag{24}
$$

The dual spherical triangle $\Delta(\xi^1, \xi^2, \xi^3)$ has edges formed by the oriented arcs $\alpha_1, \alpha_2, \alpha_3$. This construction is closely related to triangle groups generated by reflections, as considered by Fricke and Klein [6] and later by Coxeter [5]. If ξ^1, ξ^2, ξ^3 are taken as root vectors, they generate Weyl reflections r^1, r^2, r^3 in the edges of the first spherical triangle,

$$r^i : \boldsymbol{x} \to \boldsymbol{x} - 2(\boldsymbol{x} \cdot \xi^i)\xi^i \qquad i = 1, 2, 3. \tag{25}$$

The product $r^2 r^1$ generates a positive rotation with angle $2\alpha_3$ and fixpoint η^3. This rotation is an element of the adjoint representation Ad_g of SU(2) obtained by conjugation with the element g, see equation (34). The rotation is of finite order if the Coxeter condition $2\alpha_3 = 2\pi/p_3, p_3$ integer, is fulfilled. We shall come back later to the condition of finite order.

The mutual relations of the triple g_1, g_2, g_3 are determined by the angles in the two dual triangles. These relations do not change if we conjugate all three group elements with another fixed group element. To characterize the triple modulo conjugation, we turn to the Fricke characters for SU(2). Discrete systems of traces for matrices from $Sl(2, C)$ are known as Fricke characters [6]. For more recent work on Fricke characters compare Horowitz [8] and for homomorphisms $F_2 \to Sl(2, C)$ Humphreys [9]. From the exponential parametrization we get

$$\tfrac{1}{2}\,\mathrm{tr}(g_1 g_2) = (\xi^1 \cdot \xi^2) \tag{26}$$

which determines the cosine of the arc corresponding to g_3. Next we consider the matrix a with entries $a_{ij} = (\xi^i \cdot \xi^j), i, j = 1, 2, 3$. Here we get the identity

$$\begin{aligned}\det(a) &= 1 + 2(\xi^1 \cdot \xi^2)(\xi^2 \cdot \xi^3)(\xi^3 \cdot \xi^1) - (\xi^1 \cdot \xi^2)^2 - (\xi^2 \cdot \xi^3)^2 - (\xi^3 \cdot \xi^1)^2 \\ &= -\tfrac{1}{4}(\mathrm{tr}(g_2 g_1 g_3) - 2).\end{aligned} \tag{27}$$

The right-hand part of this equation, expressed by traces by use of equation (26), is due to Fricke and Klein [6]. This part is a *trace identity* which relates the trace of the commutator $K(g_2, g_1)$ to the traces of the three group elements g_1, g_2, g_3. When expressed in terms of the vectors ξ^i, it has a *geometric content* as it determines up to a factor the square of the volume of the tetrahedron spanned by ξ^1, ξ^2, ξ^3.

A homomorphism $F_2 \to \mathrm{SU}(2), \mathrm{SU}(1,1)$ is defined in terms of images of the generators of F_2, $x_1 \to g_1, x_2 \to g_2$, it implies $e \to e_{\mathrm{SU}(2)}, e_{\mathrm{SU}(1,1)}$. In the Fricke–Klein geometry it determines a triple $g_1, g_2, g_3 = (g_1 g_2)^{-1}$ and a corresponding triangle ξ^1, ξ^2, ξ^3. Now we determine the *transformations of the triangles induced by the three generators of* Φ_2. The images of ξ^1, ξ^2, ξ^3 under these generators may be linearly expressed by these vectors and by their scalar products as

$$(\zeta^1 \zeta^2 \zeta^3) = (\xi^1 \xi^2 \xi^3) D(\rho) \tag{28}$$

with the result

$$D(P) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2\epsilon(\xi^2 \cdot \xi^3) & 2\epsilon(\xi^3 \cdot \xi^1) & 1 \end{bmatrix}$$

$$D(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2\epsilon(\xi^2 \cdot \xi^3) & 1 \end{bmatrix}$$

$$D(U) = \begin{bmatrix} 2\epsilon(\xi^3 \cdot \xi^1) & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{29}$$

Here the number ϵ takes the value $\epsilon = 1$ for SU(2). The transformations of equation (29) have a simple geometric interpretation since the vectors are permuted, inverted or reflected according to equation (25). When multiplying these transformations in each step one must transform the scalar products in each step. By use of equation (26), the transformations of these scalar products become recursive trace maps. To show that these transformations represent homomorphisms from Φ_2, one must verify for the images the defining relations of the generators of Φ_2 given by Nielsen and by Magnus [12]. All three generators in equation (29) have a determinant ± 1 and preserve $\det(a)$. Hence by equation (27) *the volume of the tetrahedron spanned by* ξ^1, ξ^2, ξ^3 *is a geometric invariant under the transformations induced by* Φ_2. We shall apply these transformations to the Fibonacci system in section 7.

4. Fricke–Klein geometry of SU(1,1)

Similar results hold for $SU(1,1)$. For the exponential parametrization we shall use the basis

$$\sigma_1' = \sigma_1 \qquad \sigma_2' = -\sigma_2 \qquad \sigma_3' = \mathrm{i}\sigma_3 \tag{30}$$

of equation (16). The scalar and vector product correspond to the metric of $SO(2,1,R)$, they are

$$\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{b}) &= a_1b_1 + a_2b_2 - a_3b_3\\
(\boldsymbol{a}\times\boldsymbol{b})_1 &= (a_2b_3 - a_3b_2)\\
(\boldsymbol{a}\times\boldsymbol{b})_2 &= (a_3b_1 - a_1b_3)\\
(\boldsymbol{a}\times\boldsymbol{b})_3 &= -(a_1b_2 - a_2b_1).
\end{aligned}\tag{31}$$

These definitions yield the triple product of three vectors still in the form of a determinant,

$$\det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}. \tag{32}$$

The exponential parametrization reads

$$g = \exp\left(-\alpha\sum_{i=1}^{3}\eta_i\sigma_i'\right). \tag{33}$$

The adjoint representation Ad_g is defined by the linear transformation of the vector η' induced by conjugation,

$$\begin{aligned}
&g' \to gg'g^{-1}\\
&\eta' \to (\mathrm{Ad}_g\eta')\\
&(\mathrm{Ad}_g\eta)_i' = \sum_{j=1}^{3}(\mathrm{Ad}_g)_{ij}\eta_j'.
\end{aligned}\tag{34}$$

The adjoint representation yields a homomorphism from SU(1,1) to $SO(2,1,R\uparrow)$ which is two-to-one since $\mathrm{Ad}_g = \mathrm{Ad}_{-g}$ and *orthochronous*, that is obeys $(\mathrm{Ad}_g)_{33} \geqslant 1$.

Consider triples of group elements from SU(1,1) obeying

$$g_1 g_2 g_3 = e \tag{35}$$

associate with them triples of vectors η^1, η^2, η^3 through the exponential parametrization, and define the reciprocal triple b^1, b^2, b^3 by

$$b^i = \epsilon_{ikl}(\eta^k \times \eta^l). \tag{36}$$

For these triples we obtain the following properties. For any triple of group elements obeying equation (35), the three vectors b^i are of two types: (i) *all space-like*, $(b^i \cdot b^i) > 0$; or (ii) *all time-like*, $(b^i \cdot b^i) < 0$. If we define in these two cases

$$\begin{aligned} \xi^i &:= b^i \sqrt{\epsilon(b^i \cdot b^i)}^{\,-1} \\ \epsilon &= \pm 1 \end{aligned} \tag{37}$$

the triples ξ^1, ξ^2, ξ^3 determine points and triangles on a space-like unit hyperboloid or on a single time-like unit hyperboloid respectively. The arcs on the hyperboloids are always on intersections with a plane through the origin. We obtain the trace result

$$\tfrac{1}{2}\,\mathrm{tr}(g_1 g_2) = \epsilon(\xi^1 \cdot \xi^2). \tag{38}$$

Proof. For the first part we use the fact that elements of the adjoint representation may be generated by pairs of reflections. With the vectors b^i we associate the reflections

$$r^i : \boldsymbol{x} \to \boldsymbol{x} - 2(\boldsymbol{x} \cdot b^i)(b^i \cdot b^i)^{-1} b^i. \tag{39}$$

We require that the products $(r^1 r^2), (r^2 r^3), (r^3 r^1)$ generate elements of the *orthochronous* part of $SO(2,1,R)$. This condition yields the restriction $(b^1 \cdot b^1)(b^2 \cdot b^2) > 0$ and similar relations for all the pairs. From the action of $(r^1 r^2) \in SO(2,1,R\uparrow)$ on b^2 according to equation (39) we obtain for the (hyperbolic) cosine of twice the group parameter

$$\begin{aligned} (b^2 \cdot (r^1 r^2 b^2))(b^2 \cdot b^2)^{-1} &= \epsilon(-1 + 2(b^1 \cdot b^2)^2((b^1 \cdot b^1)(b^2 \cdot b^2))^{-1}) \\ &= \epsilon(-1 + 2(\xi^1 \cdot \xi^2)^2). \end{aligned} \tag{40}$$

If the absolute value of the expression on the left of this equation is in the range $\langle 0, 1\rangle$, it determines the cosine of twice a rotation angle. If it is in the range $\langle 1, \infty)$, it determines the hyperbolic cosine of twice a hyperbolic angle. The result given in equation (38) is obtained by an explicit computation for all representative triples. Case (ii) with $\epsilon = -1$ applies if and only if $\eta^i \cdot \eta^i = 1, |\eta^i \cdot \eta^j| < 1,\ i,j = 1,2,3,\ i \neq j$. The time-like vectors ξ^i are all found on a single hyperboloid. Case (i) with $\epsilon = 1$ applies in all other cases. □

For the matrix a with entries $a_{ij} = (\xi^i \cdot \xi^j)$ one finds

$$\begin{aligned}\det(a) &= \epsilon(1 + 2(\xi^1 \cdot \xi^2)(\xi^2 \cdot \xi^3)(\xi^3 \cdot \xi^1) - (\xi^1 \cdot \xi^2)^2 - (\xi^2 \cdot \xi^3)^2 - (\xi^3 \cdot \xi^1)^2) \\ &= -\tfrac{1}{4}(\mathrm{tr}(g_2 g_1 g_3) - 2)\end{aligned} \tag{41}$$

which is a trace identity and has a similar geometric interpretation as in the case of SU(2).

The transformations of the triangles induced by Nielsen's generators of F_2 are given again by equation (29), preserve the geometric invariant $\det(a)$ of equation (41), but have the metric of $SO(2,1,R)$ and the values $\epsilon = \pm 1$ for types (i) and (ii) respectively. The vectors ξ^i are permuted, inverted or reflected according to equation (39). We turn now to the relation between the group $SU(1,1)$ and the Schrödinger equation.

5. Transfer and scattering matrices

The one-dimensional Schrödinger equation may be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = -v(x)\psi(x) \tag{42}$$

$$v(x) = \frac{2m}{\hbar^2}(E - V(x)). \tag{43}$$

With $\psi_1(x) := \psi(x), \psi_2(x) := \mathrm{d}\psi(x)/\mathrm{d}x$ one obtains the first-order system of linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -v(x) & 0\end{bmatrix}\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix}. \tag{44}$$

This system is canonical with the Hamiltonian $H = \frac{1}{2}(\psi_1^2 + v\psi_2^2)$, but for general $v = v(x)$ it corresponds to an oscillator with time-dependent frequency. The transfer matrix defined by the equations

$$\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix} = \tilde{T}(x,x')\begin{bmatrix}\psi_1(x')\\ \psi_2(x')\end{bmatrix} \tag{45}$$

$$\tilde{T}(x,x) = e \tag{46}$$

obeys the same differential equation with respect to x as the column vector formed from $\psi_1(x), \psi_2(x)$. Since the matrix in equation (44) is traceless, the transfer matrix is unimodular and, for a real choice of the two functions, belongs to $Sl(2,R)$. For an interval with $V = 0$ and with $E = (\hbar^2/2m)k^2$, the transfer matrix may be taken as

$$\tilde{T}^0(x,x') = \begin{bmatrix}\cos k(x-x') & k^{-1}\sin k(x-x')\\ -k\sin k(x-x') & \cos k(x-x')\end{bmatrix}. \tag{47}$$

We convert from this real standing wave to a complex running wave picture by transforming the states and the transfer matrix with the matrix

$$R = \sqrt{\frac{1}{2}}\begin{bmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{bmatrix}\begin{bmatrix}\sqrt{k} & 0\\ 0 & \sqrt{k}^{-1}\end{bmatrix}. \tag{48}$$

The new form of the transfer matrix we write as $T := R\tilde{T}R^{-1}$, it is complex and belongs to the Lie group $\mathrm{SU}(1,1) = R\,\mathrm{Sl}(2,R)R^{-1}$, a complex equivalent form of $\mathrm{Sl}(2,R)$. For $V = 0$ one obtains

$$T^0(x,x') = \begin{bmatrix} \exp(ik(x-x')) & 0 \\ 0 & \exp(-ik(x-x')) \end{bmatrix}. \tag{49}$$

The general transfer matrices will now admit the description and classification given in section 2 for the elements of $\mathrm{SU}(1,1)$.

Suppose that the potential $V(x)$ is non-zero only in a finite interval say $\langle 0,c\rangle$ on R_1, and consider the transfer matrix T on this interval. Given in the interval $(-\infty,0\rangle$ the two free solutions $\exp(\pm ikx)$ with complex amplitudes, the transfer matrix determines the amplitudes of the two free solutions $\exp(\pm ik(x-c))$ in the interval $\langle c,\infty)$. It follows that the scattering matrix S, which describes scattering from the left and from the right, is determined by T. We also refer to Borland [4] for this relation. To write the explicit connection we apply to T the Gauss factorization, compare Barut and Raczka [3], given by

$$T = \begin{bmatrix} \lambda & \mu \\ \overline{\mu} & \overline{\lambda} \end{bmatrix} \tag{50}$$

$$= \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix} = \begin{bmatrix} t - rt^{-1}l & t^{-1}r \\ -lt^{-1} & t^{-1} \end{bmatrix} \tag{51}$$

which yields the relations

$$t = \overline{\lambda}^{-1} \qquad r = \overline{\lambda}^{-1}\mu \qquad l = -\overline{\lambda}^{-1}\overline{\mu}. \tag{52}$$

Here, λ is always invertible. The Gauss factors do not belong to $\mathrm{SU}(1,1)$. Now it is easy to show that

$$T\begin{bmatrix} 1 \\ l \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} \qquad T^{-1}\begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} \tag{53}$$

so that l, t are the reflection and transmission amplitudes for scattering from the left, and r, t are reflection and transmission amplitudes for scattering from the right. The scattering matrix becomes

$$S = \begin{bmatrix} t\exp(-ikc) & r\exp(-2ikc) \\ l & t\exp(-ikc) \end{bmatrix}. \tag{54}$$

This matrix is unitary due to the fact that $T \in \mathrm{SU}(1,1)$. The phase factors are required by the asymptotic form of the free states assumed in scattering, they ensure that $V = 0$ implies $S = e$. Computation of the scattering matrix for a product of transfer matrices yields the star product of S-matrices defined by Redheffer [13]. The transmission coefficient for scattering from the left is $t = \overline{\lambda}^{-1}$.

6. Fibonacci systems

The discrete Fibonacci system has the basic iterative equation

$$\phi^{n+1}(x_2) = \phi^{n-1}(x_2)\phi^n(x_2), \phi(x_1) = x_2, \phi(x_2) = x_1x_2 \tag{55}$$

which yields upon inversion

$$\phi^{n-2}(x_2^{-1}) = \phi^{n-1}(x_2)\phi^n(x_2^{-1}). \tag{56}$$

The physical model will arise by associating with the words formed from x_1, x_2 products of transfer matrices and thus determines a homomorphism $T : F_2 \to SU(1,1)$ or $F_2 \to SU(2)$. We shall first treat properties which apply to both groups. It suffices to specify the map $x_1 \to T(x_1) = T_1, x_2 \to T(x_2) = T_2$ and to use $T(w_1w_2) = T(w_1)T(w_2)$. We use the short-hand notation $T_n = T(\phi^{n-1}(x_1))$. Using this homomorphism and adding the matrix expressions corresponding to equations (55) and (56) one finds

$$T_{n+1} + T_{n-2}^{-1} = T_{n-1}(T_n + T_n^{-1}). \tag{57}$$

If now one defines three unimodular 2×2 matrices

$$(X, Y, Z)(n) = (X(n), Y(n), Z(n)) := (T_{n-2}, T_{n-1}, T_n) \tag{58}$$

one gets the recursive 12-dimensional system

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}(n+1) = \begin{pmatrix} Y \\ Z \\ Y(Z + Z^{-1}) - X^{-1} \end{pmatrix}(n). \tag{59}$$

We decompose any unimodular matrix as

$$M = S + A, A = M - \tfrac{1}{2}\operatorname{tr}(M)e \qquad \operatorname{tr}(A) = 0 \qquad S = M - A. \tag{60}$$

Note the following decoupling properties of the discrete dynamical system. The S-part of the system decouples and becomes equivalent to a system of traces. The three coefficients in the A-part couple only to the S-part. By the theorem of Nielsen, the Fibonacci system has the improper matrix invariant

$$K = x_1x_2x_1^{-1}x_2^{-1}. \tag{61}$$

For the dynamical system, it proves convenient to introduce

$$x_3 := x_1x_2. \tag{62}$$

We use the decomposition $T = S + A$ and write $T_i := T(x_i) = S_i + A_i$. Then one finds for the commutator

$$\begin{aligned} T(K^{\pm 1}) &= S \pm A \\ S &= -4S_1S_2S_3 + 2(S_1S_1 + S_2S_2 + S_3S_3) - e \\ A &= 4A_1S_2S_3 - 2(S_1A_1 - S_2A_2 + S_3A_3). \end{aligned} \tag{63}$$

In the notation of the dynamical system, one obtains for the commutator

$$T(\sigma^{n-3}(K)) = ((X+X^{-1})(n) - (Y+Y^{-1})(n)(Z+Z^{-1})(n))X^{\mp 1}(n) \\ + (Y+Y^{-1})(n)Y^{\pm 1}(n) + (Z+Z^{-1})(n)Z^{\mp 1}(n) - e. \tag{64}$$

Again, this commutator decouples into an S-part and an A-part whose coefficients couple only to the S-part. In view of equation (60), the S-part of equation (63) relates the traces. It can be interpreted in terms of the Fricke–Klein geometry of sections 3 and 4, equations (27) and (41) and leads to a *geometric invariant* of the system. The A-part has alternating signs due to the improper invariance.

We turn to $\mathrm{SU}(1,1)$ to illustrate the decoupling of the full system and write down the iterative equations for the diagonal element λ of the transfer matrix and its improper invariant:

$$\begin{pmatrix} \lambda(X) \\ \lambda(Y) \\ \lambda(Z) \end{pmatrix}(n+1) = \begin{pmatrix} \lambda(Y) \\ \lambda(Z) \\ (\lambda(Z)+\overline{\lambda}(Z))\lambda(Y) - \overline{\lambda}(X) \end{pmatrix}(n)$$

$$\lambda(K) = ((\lambda+\overline{\lambda})(X) - (\lambda+\overline{\lambda})(Y)(\lambda+\overline{\lambda})(Z))\overline{\lambda}(X) \\ + (\lambda+\overline{\lambda})(Y)\lambda(Y) + \overline{\lambda}(Z)(\lambda+\overline{\lambda})(Z) - 1. \tag{65}$$

The improper invariant part transforms in the iteration according to

$$\lambda(K) \to \overline{\lambda}(K) \to \lambda(K) \ldots .$$

From the map equation (54) to the scattering matrix, the diagonal element λ determines the forward scattering from the Fibonacci string.

In the discrete Fibonacci system, the transfer matrix is obtained on chains of length increasing with the Fibonacci numbers. We now show that the continuous Schrödinger equation for the transfer matrix can be rewritten as a continuous 12-dimensional dynamical system such that the discrete dynamical system appears in an exact relation to the continuous one. To this end we write the equation of motion for the transfer matrix in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}T(x,x') = W(x)T(x,x') \qquad W(x) = R\begin{bmatrix} 0 & 1 \\ -v(x) & 0 \end{bmatrix}R^{-1} \tag{66}$$

$$T(x,x) = e. \tag{67}$$

We take the two intervals on the line with length $1, \tau$ respectively. If these intervals carry fixed values of the potential, and if the sequence of intervals is generated through equation (55), this sequence yields the symmetry

$$V(x+\tau^n) = V(x) \qquad 0 \leqslant x \leqslant \tau^{n-1}. \tag{68}$$

The same symmetry applies to $v(x)$ and to the matrix $W(x)$. For the transfer matrix it follows from this symmetry and the equations of motion equation (66) that

$$T(\tau^{n+1}, \tau^n) = T(\tau^{n-1}, 0). \tag{69}$$

Let us introduce a new real variable z and define

$$\begin{aligned} X(z) &= T(\tau^2 z, \tau z) \\ Y(z) &= T(\tau z, 0) \\ Z(z) &= T(\tau^2 z, 0) = X(z)Y(z). \end{aligned} \tag{70}$$

Using the equations of motion for $T(x,x')$ and its inverse $T^{-1}(x,x') = T(x',x)$, one obtains for the three matrices X, Y, Z the equations of motion of a continuous dynamical system:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}X(z) &= \tau^2 W(\tau^2 z)X(z) - X(z)\tau W(\tau z) \\ \frac{\mathrm{d}}{\mathrm{d}z}Y(z) &= \tau W(\tau z)Y(z) \\ \frac{\mathrm{d}}{\mathrm{d}z}Z(z) &= \tau^2 W(\tau^2 z)Z(z). \end{aligned} \tag{71}$$

For the discrete values $z = \tau^n$, it follows from equation (68) that

$$\begin{aligned} X(\tau^n) &= T(\tau^{n+2}, \tau^{n+1}) = T(\tau^n, 0) \\ Y(\tau^n) &= T(\tau^{n+1}, 0) \\ Z(\tau^n) &= T(\tau^{n+2}, 0) = X(\tau^n)Y(\tau^n). \end{aligned} \tag{72}$$

For these value of z, the three matrices are seen to coincide with the matrices $X(n+1), Y(n+1), Z(n+1)$ of the discrete dynamical Fibonacci matrix system. We conclude that the matrices of the continuous system must run, for $z = \tau^n$, through the values of the discrete system. In view of the restriction to SU(1,1), both systems have nine (group) parameters. The discrete dynamical system has the improper commutator invariant which provides a three-dimensional (Poincare-like) section for the continuous system. Note that we get an *exact discrete dynamics* on the three-dimensional sections.

Finally we consider a particular form of the transfer matrices for the Fibonacci chain. We assume that, on both cells, the same potential with transfer matrix $T(\epsilon, 0)$ is followed by two intervals with potential equal to zero and with transfer matrices $T^0(1,\epsilon), T^0(\tau,\epsilon)$. The commutator becomes

$$K(T^0(1,\epsilon)T, T^0(\tau,\epsilon)T)) = T^0(1,\epsilon)K(T, T^0(\tau-1,0))(T^0(\epsilon,1) \tag{73}$$

and so is equivalent to the commutator of the free transfer matrix for an interval of length $\tau - 1$ with the transfer matrix for the potential. From equation (49) the first matrix obeys $T^0(\tau-1,0) = e$ for $k(\tau-1) = m\pi, m \in Z$. At these *periodic points* with respect to the wavenumber k, the commutator has the value $K = e$ and the trace $\frac{1}{2}\operatorname{tr}(K) = 1$. For all other values of k we have, in the classification of equation (19), $T^0(\tau-1,0) \in C_+^-$ so that from equation (19) $K \in C_0^0$ or C_+^+. It follows that $\frac{1}{2}\operatorname{tr}(K) \geqslant 1$ and, for differentiable dependence of T on k, that *the periodic zeros of the invariant are quadratic minima*. These properties can be seen in the computed values of the invariant $I = \frac{1}{4}(\operatorname{tr}(K) - 2)$ for the special case where T represents a δ-potential of strength q and hence belongs to the class type C_+^0. For this case, studied in more detail in Baake *et al* [1], the table in equation (19) shows that there are no points $K = e$ except the ones periodic in k. The explicit value of the invariant becomes

$$I = (\omega \sin k(\tau - 1))^2 \qquad \omega = q/2k. \tag{74}$$

The implications for the electron states and spectra on this and on generalized Fibonacci systems are treated in Baake *et al* [1].

7. The Fibonacci system in Fricke–Klein geometry

An alternative geometric representation of the Fibonacci system is offered by use of the Fricke–Klein geometry. We start with the Fibonacci system represented in SU(2). For three group elements $g_1, g_2, g_3 = g_1 g_2$ we have $g_1 g_2 g_3^{-1} = e$. We wish to use for this new triple the old dual vectors ξ^1, ξ^2, ξ^3 given in section 3. For this purpose we keep the same dual vectors but reverse the orientation of the arc between ξ^1 and ξ^2. With this interpretation we get a correspondence

$$T_{n+1} = T_{n-1} T_n \to g_{n+1} = g_{n-1} g_n \to \Delta(\xi^{n-1} \xi^n \xi^{n+1}). \tag{75}$$

The Fibonacci system described by the powers ϕ^n now generates a sequence of dual triangles on the sphere, spanned by $\xi^{n-1}, \xi^n, \xi^{n+1}$. This sequence of vectors is determined by writing $\phi = P \circ U$. By combining the transformations $D(P), D(U)$ from equation (29) and rearranging one finds for the present triples the transformation law

$$(\xi^{n-1} \xi^n \xi^{n+1}) = (\xi^{n-2} \xi^{n-1} \xi^n) \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2\epsilon(\xi^n \cdot \xi^{n-2}) \end{bmatrix} \qquad n \geqslant 4 \tag{76}$$

with initial values $\xi^2, \xi^3, \xi^4 = \xi^1$. From this equation one determines for the scalar products the rule

$$\begin{aligned} &\epsilon(\xi^{n+1} \cdot \xi^{n-1}) = -\epsilon(\xi^{n-2} \cdot \xi^{n-1}) + (\xi^n \cdot \xi^{n-2})(\xi^n \cdot \xi^{n-1}) \\ &(\xi^{n+1} \cdot \xi^n) = (\xi^n \cdot \xi^{n-2}) \qquad n \geqslant 4. \end{aligned} \tag{77}$$

which for $\epsilon = 1$ is equivalent to the recursive trace map of SU(2). In the new equations for the Fibonacci matrix system, the recursive computation employs only the mutual relations of matrices, independently of conjugation transformations applied to the initial group elements. The volume of the tetrahedron spanned by the three vectors is a *geometric invariant* and is, through equation (27), directly related to the trace of the commutator in the matrix system. A special case arises if the two initial group elements g_1, g_2 belong to a finite subgroup H of SU(2). Clearly then the orbits under the Fibonacci system must become *periodic*, with the period bounded by the order of H.

For the Fibonacci system in the Fricke–Klein geometry of SU(1, 1) we find, under the conditions given in section 4, exactly the same relations equations (76) and (77), but with the scalar products now corresponding to SU$(1,1)$. The types (i) and (ii) have $\epsilon = \pm 1$. Equation (77) together with equation (38) yields the recursive trace map. There are *two different types of orbits*: (i) if all three vectors ξ^i are space-like, the system of triangles runs on the single unit hyperboloid in the space-like region; (ii) if all three vectors ξ^i are time-like, the system runs on a single unit hyperboloid in the time-like region. In this case, the vectors η^i are all space-like. The system has the geometric invariant given by equation (41). If the two initial group elements belong to one of the discrete subgroups specified by Fricke and Klein [6], we expect particular properties of the orbits under the Fibonacci system. Again, the propagation of the triangles is independent of conjugation transformations. The geometric transformations for generalized Fibonacci systems and in fact for any automorphism from Φ_2 may be generated from the transformations equation (29).

Acknowledgments

This work presents an algebraic part of studies on quasiperiodic systems, done in cooperation with M Baake, J Garcia Escudero and D Joseph, and supported by the Deutsche Forschungsgemeinschaft.

References

[1] Baake M, Joseph D and Kramer P 1992 *Phys. Lett.* **186A** 199–208
[2] Bargmann V 1961 *Commun. Pure Appl. Math.* **14** 187
[3] Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: Polish Scientific Publishers)
[4] Borland E 1966 *Mathematical Physics in One Dimension* ed E H Lieb and D C Mattis (New York: Academic)
[5] Coxeter H M S 1973 *Regular Polytopes* 3rd edn (New York: Dover)
[6] Fricke R and Klein F 1897 *Vorlesungen über die Theorie der Automorphen Funktionen* vol 1 (Leipzig: Teubner)
[7] Holzer M 1988 *Phys. Rev.* B **38** 5756–9
[8] Horowitz R D 1975 *Trans. Am. Math. Soc.* **208** 41–50
[9] Humphreys St P 1992 *Math. Z.* **209** 131–52
[10] Kohmoto M, Kadanoff L P and Tang C 1983 *Phys. Rev. Lett.* **50** 1870
[11] Lothaire M 1983 *Combinatorics on Words* (Reading, MA: Addison-Wesley)
[12] Magnus W, Karrass A and Solitar D 1976 *Combinatorial Group Theory* 2nd edn (New York: Dover)
[13] Redheffer R 1961 *Modern Mathematics for the Engineer* ed E F Beckenbach and M R Hestenes (New York: McGrawHill) pp 282–456
[14] Sutherland B 1986 *Phys. Rev. Lett.* **57** 770–3